Announcements

1) Fix for last example, change sine to cosine in answer.

2) Problems to write up and turn in from Webwork. #'s 8, 11 (Thursday)

Recall: Wanted to show

$$\frac{d}{dx}(\sin(x)) = \cos(x).$$

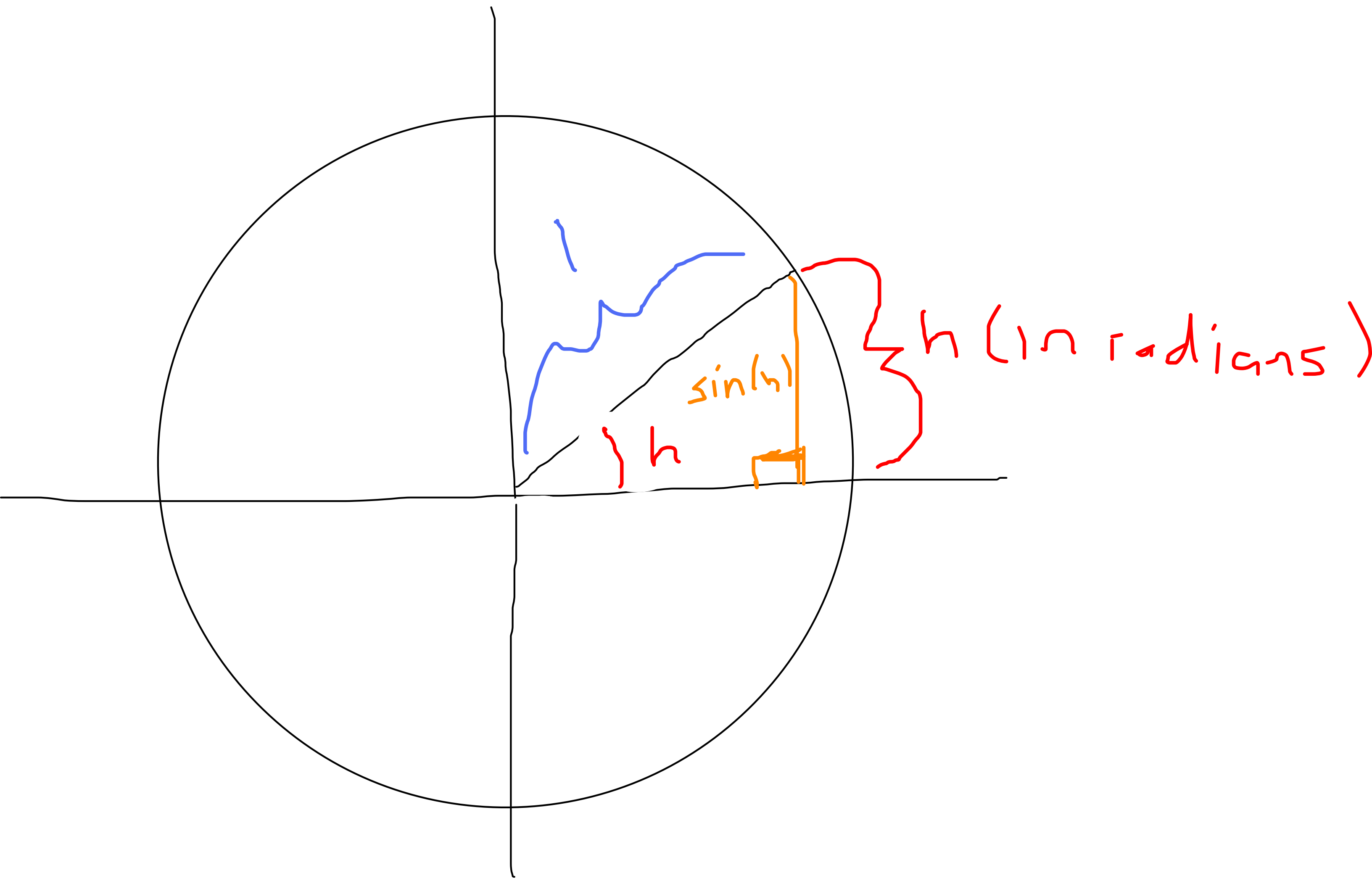
Reduced the problem to showing

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

$$\frac{0}{0} = \text{more work!}$$

Use Squeeze Theorem

Picture (unit circle)



$$\sin h = \frac{opp}{hyp} = \frac{opp}{1}$$

length of side opposite h = sin(h)

Since the arc is bigger,

$$\boxed{\sin(h) \le h}$$

Divide by h.

$$\frac{\sin(h)}{h} \le 1$$

You can show, using more involved geometric calculations, that

$$\boxed{\cos(h) \le \frac{\sin(h)}{h}}$$

We then have

$$\cos(h) \le \frac{\sin(h)}{h} \le 1$$

Take limit as $h \to 0$, use Squeeze Theorem to obtain

$$\underset{\underset{\displaystyle 1}{\parallel}}{\lim_{h\to 0} \cos(h)} \le \lim_{h \to 0} \frac{\sin(h)}{h} \le 1$$

So

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

Note: You will have to remember that

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

Hence, $\boxed{\frac{d}{dx}(\sin(x)) = \cos(x)}$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(x + \frac{\pi}{2}\right)\right)$$

$$= \cos\left(x + \frac{\pi}{2}\right) \cdot 1$$

(chain rule)

$$= \cos\left(x + \frac{\pi}{2}\right)$$

$$= \boxed{-\sin(x)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= \sin(x)\,(\cos(x))^{-1}$$

$$= \sin(x)\sec(x).$$

First find $\frac{d}{dx}\left(\sec(x)\right)$

$$= \frac{d}{dx}\left((\cos(x))^{-1}\right)$$

$$= (-1)\,(\cos(x))^{-2}\,(-\sin(x))$$

(chain rule)

$$= \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)}$$

$$= \boxed{\tan(x)\sec(x)}$$

Now use product rule to find the derivative of $\tan(x)$,

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}(\sin(x)\sec(x))$$

$$= \overbrace{\cos(x)}^{\frac{d}{dx}(\sin(x))}\sec(x)$$

$$+ \sin(x)\underbrace{\sec(x)\tan(x)}_{\frac{d}{dx}(\sec(x))}$$

$$= 1 + \tan^2(x)$$

$$= \boxed{\sec^2(x)}$$

Other trig derivatives

$$\boxed{\frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)}$$

$$\boxed{\frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)}$$

Note: All derivatives of "co" functions all have negative signs

Example 1: $f(x) = x^4 \sin(x)$

Find $f'(x)$.

Use product rule.

$$f'(x) = x^4 (\sin(x))' + \sin(x)(x^4)'$$

$$= \boxed{x^4 \cos(x) + 4x^3 \sin(x)}$$

Example 2: $f(\theta) = \csc(\sec(\sqrt[5]{\theta}))$

Find $\frac{df}{d\theta}$.

Let $g(\theta) = \sec(\sqrt[5]{\theta})$

Use chain rule.

$$\frac{df}{d\theta} = \left(\frac{d}{d\theta}\csc\right)(g(\theta)) \cdot \frac{dg}{d\theta}$$

$$= -\csc(g(\theta)) \cdot \cot(g(\theta)) \cdot \frac{dg}{d\theta}$$

$$= -\csc(\sec(\sqrt[5]{\theta}))\cot(\sec(\sqrt[5]{\theta})) \cdot dg/d\theta.$$

To find $\frac{dg}{d\theta}$, use the chain rule again.

$$\frac{dg}{d\theta} = \left(\frac{d}{d\theta}\sec\right)(\sqrt[5]{\theta}) \cdot \frac{d}{d\theta}(\sqrt[5]{\theta})$$

$$= \sec(\sqrt[5]{\theta})\tan(\sqrt[5]{\theta})\frac{d}{d\theta}(\sqrt[5]{\theta})$$

$$= \sec(\sqrt[5]{\theta})\tan(\sqrt[5]{\theta})\frac{d}{d\theta}(\theta^{1/5})$$

$$= \sec(\sqrt[5]{\theta})\tan(\sqrt[5]{\theta})\left(\frac{1}{5}\right)\theta^{-4/5}$$

The end result is

$$\frac{df}{d\theta} = -\csc\left(\sec\left(\sqrt[5]{\theta}\right)\right)\cot\left(\sec\left(\sqrt[5]{\theta}\right)\right) \cdot \sec\left(\sqrt[5]{\theta}\right)\tan\left(\sqrt[5]{\theta}\right) \cdot \frac{1}{5} \cdot \theta^{-4/5}$$

Example 3.

$$\lim_{\theta \to 0} \frac{\sin(15\theta)}{21\theta}$$

Note:

$$\lim_{\theta \to 0} \frac{\sin(15\theta)}{21\theta}$$

$$= \frac{1}{21} \lim_{\theta \to 0} \frac{\sin(15\theta)}{\theta}$$

$$= \frac{1}{21} \lim_{\theta \to 0} \frac{15\sin(15\theta)}{15\theta}$$

$$= \frac{15}{21} \lim_{\theta \to 0} \frac{\sin(15\theta)}{15\theta}$$

$$= \frac{15}{21} \cdot 1 = \boxed{\frac{15}{21}}$$

Note: $$\lim_{x \to 0} \frac{\sin(u)}{u}$$

where u is some function of x with $\lim_{x \to 0} u(x) = 0$

is always one.

To conclude this, you need what's in the denominator to match **exactly** with what's in the numerator

Example 4: Find $\lim\limits_{x \to 0^+} \frac{\sin(x)}{\sqrt{x}}$.

$$\frac{\sin(x)}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}\,\sin(x)}{x}$$

So $\lim\limits_{x \to 0^+} \frac{\sin(x)}{\sqrt{x}}$

$$= \lim_{x \to 0^+} \left(\sqrt{x} \cdot \frac{\sin(x)}{x} \right)$$

$$= \lim_{x \to 0^+} \sqrt{x} \cdot \lim_{x \to 0^+} \frac{\sin x}{x}$$

$$= 0 \cdot 1 = \boxed{0}$$

However: If m and n are nonzero real numbers,

$$\boxed{\lim_{\theta \to 0} \frac{\sin(m\theta)}{n\theta} = \frac{m}{n}}$$

Example 5:

$$\lim_{\theta \to 0} \frac{\tan(\theta)}{6\theta - \sin(\theta)}$$

You know: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Make this limit appear by division. Divide top and bottom by θ.

We get

$$\frac{\tan\theta}{6\theta - \sin\theta} \cdot \frac{1/\theta}{1/\theta} = 1$$

$$= \frac{\frac{\tan\theta}{\theta}}{\frac{6\theta}{\theta} - \frac{\sin\theta}{\theta}}$$

$$= \frac{\frac{\tan(\theta)}{\theta}}{6 - \frac{\sin\theta}{\theta}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\theta}}{6 - \frac{\sin\theta}{\theta}}$$

This equals

$$\frac{\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}{6-\dfrac{\sin\theta}{\theta}}.$$

Now take limit as $\theta \to 0$.

$$\lim_{\theta\to 0}\frac{\tan\theta}{6\theta-\sin(\theta)}$$

$$=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}{6-\dfrac{\sin\theta}{\theta}}$$

$$=\frac{\overset{\swarrow 1}{\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}}\cdot\lim_{\theta\to 0}\dfrac{1}{\cos\theta}=1}{6-\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1}=\boxed{\frac{1}{5}}$$

Implicit Differentiation

(Section 2.7)

Circle

Picture

Idea: Regard y as an "implicit" (i.e. you can't solve for y in terms of x) function of x.

Example 5.